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Replacing a Double Integral with a Single Integral*

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1. INTRODUCTION

The purpose of this paper is to determine situations in which an integral over a region can be replaced by an integral over the boundary of the region. The motivation here is that, when this is possible, one can perhaps get away with fewer function-evaluations in applying a quadrature formula. Furthermore, unless the region is of a very simple nature, the selection of nodes can be accomplished more judiciously for the resulting one-dimensional integral than for the original integral.

In Section 2 we make the hypothesis that the integrand satisfies a second-order linear partial differential equation. Our principal result, Theorem 2.1, is then presented with corollaries and examples. In an attempt to remove the above hypothesis, we examine in Section 3 a special case of Theorem 2.1 and proffer an application to a common but troublesome integral. Numerical data pertaining to this integral is presented in Section 4.

Throughout this paper, the following notation and (stronger than necessary) hypotheses are to be understood. S is the closure of a simply-connected bounded open set in the plane. T is the boundary of S. T is assumed to be a simple closed rectifiable curve. All functions are sufficiently smooth to enable the manipulations. ($C^2(S)$ will be more than sufficient.)

2. THE REPRESENTATION

Let L be the partial differential operator given by

$$Lu = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu, \tag{2.1}$$

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where $A, B, C \in C^2(S)$, $D, E \in C^1(S)$, $F \in C(S)$. We denote the adjoint operator by M:

$$Mv = (Av)_{xx} + (2Bv)_{xy} + (Cv)_{yy} - (Dv)_x - (Ev)_y + Fv. \tag{2.2}$$

Our main result follows.

THEOREM 2.1. *Let L and M be as in* (2.1) *and* (2.2). *Define $p(x, y)$ and $q(x, y)$ by*

$$p = Av_x + A_x v + Bv_y + B_y v - Dv, \tag{2.3}$$

$$q = Bv_x + B_x v + Cv_y + C_y v - Ev. \tag{2.4}$$

Suppose $Mv = 1$ on S. Then for all u satisfying $Lu = g$ on S, we have the identity:

$$\iint_S u \, dx \, dy = I_1 + I_2 + I_3, \tag{2.5}$$

where

$$I_1 = \iint_S vg \, dx \, dy,$$

$$I_2 = \int_T u(-q \, dx + p \, dy),$$

$$I_3 = \int_T v[(Bu_x + Cu_y) \, dx + (-Au_x - Bu_y) \, dy].$$

Proof. Define functions $r(x, y)$ and $s(x, y)$ by

$$r = -vAu_x - vBu_y + uAv_x + uA_x v + uBv_y + uB_y v - Duv,$$
$$s = -vBu_x - vCu_y + uBv_x + uB_x v + uCv_y + uC_y v - Euv.$$

The well-known Green–Lagrange Identity [1, p. 120], which may be verified in our setting by straightforward differentiation, states that

$$u(Mv) - v(Lu) = r_x + s_y$$

for all $u, v \in C^2(S)$. Since $Mv = 1$ and $Lu = g$, we obtain, upon applying Green's Theorem,

$$\begin{aligned}\iint_S u \, dx \, dy &= \iint_S u(Mv) \, dx \, dy \\ &= \iint_S v(Lu) \, dx \, dy + \iint_S (r_x + s_y) \, dx \, dy \\ &= \iint_S vg \, dx \, dy + \int_T (-s \, dx + r \, dy).\end{aligned}$$

This last line integral around T contains 14 terms (involving u and v and the coefficients of L), which in (2.5) we have segregated for future convenience into I_2 and I_3. ∎

Let us examine the representation (2.5) a bit more closely. I_1 is a double integral, which is exactly what we are trying to avoid; however, in applications it will generally be the case that $g \equiv 0$, whence $I_1 = 0$. I_2 requires knowing v in order to compute p and q; determination of v may or may not be difficult, but there is compensation in that one choice of v will then work for all u solving $Lu = g$. (The same can be said, incidentally, about I_1 in the case $g \not\equiv 0$: the evaluation of I_1 is a one-shot operation which can then be used for all solutions u of $Lu = g$.) I_3 involves u_x and u_y instead of u, thus generating another source of labor and error; the corollaries and examples below illustrate how to annihilate I_3, the procedure varying with the classification (elliptic, parabolic, hyperbolic) of L.

To rid the representation (2.5) of I_3, we first assume that the necessary transformation has already been performed to render L into its canonical form. In the elliptic case, a Dirichlet problem is to be solved for v; in the parabolic and hyperbolic cases, we restrict attention to rectangles $S = [a, b] \times [c, d]$ but require less of v.

COROLLARY 2.2 (Elliptic case: $A = C = 1$ and $B = 0$). *If v solves the problem*

$$\begin{aligned} v_{xx} + v_{yy} - (Dv)_x - (Ev)_y + Fv &= 1 \qquad \text{on } S, \\ v &= 0 \qquad \text{on } T, \end{aligned} \tag{2.6}$$

then

$$\iint_S u \, dx \, dy = I_1 + \int_T u(-v_y \, dx + v_x \, dy) \tag{2.7}$$

for all solutions u of

$$u_{xx} + u_{yy} + Du_x + Eu_y + Fu = g. \tag{2.8}$$

EXAMPLE 2.3 (*the reduced wave equation on a disc*). Let $Lu = u_{xx} + u_{yy} + \lambda^2 u$ on $S = \{(x, y)\colon x^2 + y^2 \leqslant R^2\}$. (We assume that $\lambda \neq 0$ and that λR is not a zero of the Bessel function J_0.) If we define v in polar coordinates by $v(r, \theta) = \lambda^{-2}[1 - J_0(\lambda r)]/J_0(\lambda R)$, we find that v vanishes on T and that $Mv = v_{rr} + r^{-1}v_r + r^{-2}v_{\theta\theta} + \lambda^2 v = 1$. (2.7) now leads to

$$\iint_S u \, dx \, dy = -\frac{RJ_0'(\lambda R)}{\lambda J_0(\lambda R)} \int_0^{2\pi} u(R, \theta) \, d\theta$$

for all solutions of $Lu = 0$ on S.

COROLLARY 2.4 (Parabolic case: $A = 1$ and $B = C = 0$). *If v solves the problem*

$$\begin{aligned} v_{xx} - (Dv)_x - (Ev)_y + Fv &= 1 \qquad \text{on} \quad S = [a, b] \times [c, d], \\ v(a, y) = v(b, y) &= 0 \qquad \text{for all} \quad y \in [c, d], \end{aligned} \tag{2.9}$$

then

$$\iint_S u\,dx\,dy = I_1 + \int_T u(Ev\,dx + v_x\,dy) \tag{2.10}$$

for all solutions u of

$$u_{xx} + Du_x + Eu_y + Fu = g. \tag{2.11}$$

EXAMPLE 2.5 (*the heat equation on a rectangle*). Let $Lu = u_{xx} - \lambda u_y$ on $S = [a, b] \times [c, d]$. The function $v(x, y) = (x - a)(x - b)/2$ satisfies $Mv = v_{xx} + \lambda v_y = 1$ and vanishes on the vertical sides, as required by (2.9). (2.10) now yields

$$\begin{aligned} \iint_S u\,dx\,dy &= \frac{\lambda}{2} \int_a^b (x - a)(x - b)[u(x, d) - u(x, c)]\,dx \\ &\quad + \frac{b - a}{2} \int_c^d [u(b, y) + u(a, y)]\,dy \end{aligned}$$

for all solutions of $Lu = 0$ on S.

COROLLARY 2.6 (Hyperbolic case: $B = 1$ and $A = C = 0$). *If v solves the problem*

$$v_{xy} - (Dv)_x - (Ev)_y + Fv = 1 \tag{2.12}$$

on $S = [a, b] \times [c, d]$, then

$$\iint_S u\,dx\,dy = -uv\,\Big|_{x=a,y=c}^{x=b,y=d} + I_1 + \int_T u[(Ev - v_x)\,dx + (v_y - Dv)\,dy] \tag{2.13}$$

for all solutions u of

$$u_{xy} + Du_x + Eu_y + Fu = g. \tag{2.14}$$

EXAMPLE 2.7 (*the Darboux equation on a rectangle*). Let $Lu = u_{xy} + \lambda(x + y)^{-1}(u_x + u_y)$, where $\lambda > 1$ and, to keep the analysis simple, we assume that the rectangle $S = [a, b] \times [c, d]$ does not intersect the line $x + y = 0$. A solution of the adjoint equation,

$$Mv = v_{xy} - \lambda(x + y)^{-1}(v_x + v_y) + 2\lambda(x + y)^{-2}v = 1,$$

is given by the function $v(x, y) = (x + y)^2/(2 - 2\lambda)$, and (2.13) now says

$$\begin{aligned}\iint_S u(x, y)\, dx\, dy = \frac{2-\lambda}{2-2\lambda}\Big[\int_a^b (x + d)\, u(x, d)\, dx - \int_a^b (x + c)\, u(x, c)\, dx \\ + \int_c^d (b + y)\, u(b, y)\, dy - \int_c^d (a + y)\, u(a, y)\, dy\Big] \\ + \frac{1}{2\lambda - 2}\,[(b + d)^2\, u(b, d) - (b + c)^2\, u(b, c) \\ - (a + d)^2\, u(a, d) + (a + c)^2\, u(a, c)]\end{aligned}$$

for all solutions of $Lu = 0$ on S.

3. General Integrands

In the preceding section, the objective was to develop a one-dimensional integration formula for $\iint_S u\, dx\, dy$, valid for all solutions u of a partial differential equation $Lu = g$, in which L and g have been pre-specified. We now turn to the more practical problem: can we develop such dimension-reducing formulas for arbitrary functions u, i.e., when no L and g are given? The answer will be in the affirmative provided that, given u, we can come up with an operator L, tailored to u, in such a way that the difficulties in finding a function v for which $Mv = 1$ are surmountable. To this end, we first restate Theorem 2.1 in the context of first-order operators.

Corollary 3.1. *Let v satisfy the equation*

$$-(Dv)_x - (Ev)_y + Fu = 1$$

on S. Then the identity

$$\iint_S u\, dx\, dy = \int_T uv(E\, dx - D\, dy) \tag{3.1}$$

obtains for all solutions u of

$$Du_x + Eu_y + Fu = 0.$$

Proof. This is Theorem 2.1 with $A = B = C = 0$. ∎

Now let u be a given bivariate function. An appropriate choice of L, in the light of corollary 3.1, is the operator $L = u_y(\partial/\partial x) - u_x(\partial/\partial y)$. For then $Lu = 0$, and so if v satisfies

$$Mv = -u_y v_x + u_x v_y = 1,$$

then (3.1) gives

$$\iint_S u\, dx\, dy = \int_T uv(-u_x\, dx - u_y\, dy).$$

If S is a rectangle situated parallel to the axes, an integration by parts produces

$$\iint_S u\, dx\, dy = \tfrac{1}{2} \int_T u^2(v_x\, dx + v_y\, dy). \tag{3.2}$$

In fact, (3.2) holds even when S is not rectangular, as is seen from the case $\phi(t) = t^2/2$ of the following elementary result.

THEOREM 3.2. *Let $u \in C^1(S)$, let ϕ be a differentiable univariate function, and suppose $v \in C^2(S)$ satisfies*

$$(-u_y v_x + u_x v_y)\, \phi'(u) = u. \tag{3.3}$$

Then

$$\iint_S u\, dx\, dy = \int_T \phi(u)\, dv. \tag{3.4}$$

Proof. Using Green's Theorem,

$$\begin{aligned}
&\int_T [\phi(u)\, v_x\, dx + \phi(u)\, v_y\, dy] \\
&\quad = \iint_S [\phi'(u)\, u_x v_y + \phi(u)\, v_{yx} - \phi'(u)\, u_y v_x - \phi(u)\, v_{xy}]\, dx\, dy \\
&\quad = \iint_S u\, dx\, dy. \quad \blacksquare
\end{aligned}$$

Utilization of Theorem 3.2 is seriously limited by the difficulty of locating a function v satisfying (3.3). (This may even be impossible: if, for example, grad $u = 0$ at a point in S at which $u \neq 0$.) If v cannot be found analytically, one could attempt an approximation of v by numerical methods, but this of course entails considerably greater expense than the direct numerical approximation of $\iint_S u\, dx\, dy$.

In a more optimistic vein, there are numerous instances of integrands which respond favorably to Theorem 3.2, as the following examples are meant to illustrate. Furthermore, it should be emphasized that, once v has been constructed, (3.4) can be exploited to integrate u over a virtually unlimited selection of regions S.

EXAMPLE 3.3. Let $u = \lambda x + \mu y$, where λ and μ are constants. Corresponding to $\phi_1(t) = t^2/2$, choose $v_1 = (\mu x - \lambda y)/(\lambda^2 + \mu^2)$; for $\phi_2(t) = t$, choose $v_2 = [-\lambda\mu x^2 + (\lambda^2 - \mu^2)xy + \lambda\mu y^2]/(\lambda^2 + \mu^2)$. These pairings honor (3.3), and (3.4) gives

$$\begin{aligned}\iint_S u\, dx\, dy &= \tfrac{1}{2} \int_T u^2\, dv_1 \\ &= \int_T u\, dv_2 .\end{aligned}$$

EXAMPLE 3.4. Let $u = x^2 + y^2$, and assume $(0, 0) \notin S$. Using $\phi(t) = t$, we solve

$$-2yv_x + 2xv_y = x^2 + y^2$$

and obtain, for example,

$$v(x, y) = -\tfrac{1}{2}(x^2 + y^2)a(x, y), \tag{3.5}$$

where $a(x, y)$ is a continuous determination of the inverse tangent of x/y. (Simple connectivity and exclusion of (0, 0) enable this.) Then

$$\iint_S u\, dx\, dy = \int_T u\, dv.$$

Before proceeding to the examples which are the real reason for this paper, we state a lemma which greatly aids the quest for v in many instances. It says essentially that if v is known for a given u, it is easy to find the v for a univariate function of u. The proof is by direct calculation and will be omitted.

LEMMA 3.5. *Let θ and ψ be univariate functions, and let u, v, w be bivariate functions for which $\partial(u, w)/\partial(x, y) = \theta(u)$ and $v = \psi(u)w/\psi'(u)\theta(u)$. Then $\partial(\psi(u), v)/\partial(x, y) = \psi(u)$.*

EXAMPLE 3.6. We will apply (3.4) to the evaluation of $\iint_S \exp(-x^2 - y^2)\, dx\, dy$, where $(0, 0) \notin S$. (3.5) states that

$$\frac{\partial(x^2 + y^2, -\frac{1}{2}(x^2 + y^2)\, a(x, y))}{\partial(x, y)} = x^2 + y^2,$$

and it follows from Lemma 3.5, with $\theta(t) = t$ and $\psi(t) = \exp(-t)$, that

$$\frac{\partial(\exp(-x^2 - y^2), v)}{\partial(x, y)} = \exp(-x^2 - y^2),$$

where $v = \frac{1}{2}a(x, y)$. With this choice of v, Theorem 3.2 gives

$$\iint_S \exp(-x^2 - y^2)\, dx\, dy = \int_T \exp(-x^2 - y^2)\, dv. \tag{3.6}$$

Numerical data involving the representation (3.6) for several regions S is presented in the next section. We remark here that the restriction $(0, 0) \notin S$ can be avoided by removing a narrow corridor R containing $(0, 0)$ and connecting $(0, 0)$ to T; the error so introduced is controlled by

$$\left| \iint_S u - \iint_{S \setminus R} u \right| \leqslant \max_R |u| \cdot (\text{area of } R). \tag{3.7}$$

4. Numerical Results

How valuable is the representation (3.6)? In this section we will examine it on several regions S for comparison with standard techniques.

A composite trapezoidal approximation to the Stieltjes integral $\int_T u\, dv$ will be employed: upon parametrizing T by t on $[0, 1]$, letting $t_i = i/N$ ($i = 0, 1,..., N$), defining $u_i = u(x(t_i), y(t_i))$ and v_i similarly, we set

$$K_N \equiv \sum_{i=0}^{N-1} (v_{i+1} - v_i)(u_{i+1} + u_i)/2 \tag{4.1}$$

$$\approx \int_T u\, dv.$$

Thus, our method is

$$\iint_S \exp(-x^2 - y^2)\, dx\, dy \approx K_N, \tag{4.2}$$

where K_N is given by (4.1) with

$$\begin{aligned} u &= \exp(-x^2 - y^2), \\ v &= \tfrac{1}{2}a(x, y). \end{aligned} \tag{4.3}$$

The results reported below could no doubt be sharpened by means of an extrapolation process on (4.1) or by invoking Gaussian quadrature in the first place, but we prefer to use unsophisticated quadrature techniques here in order to make the comparisons clearer. One could also economize immensely in using (4.2) by performing some preliminary analysis to select an efficient parametrization of T or, equivalently, by forsaking equal spacing of the t_i ; in the examples below we have resisted the urge to do this, although the temptation was especially strong in Example 4.2.

EXAMPLE 4.1. Let $S = [1, 3] \times [1, 2]$. Since the integral $\int_1^2\int_1^3 \exp(-x^2 - y^2)\, dx\, dy$ readily factors into two one-dimensional integrals, our method (4.2) is barely competitive with standard procedures. Listed below is the value of K_N for various N:

$$
\begin{array}{ll}
N = 8 & K_N = .202225(-1) \\
N = 40 & K_N = .189045(-1) \\
N = 200 & K_N = .188548(-1) \\
N = 1000 & K_N = .188525(-1)
\end{array}
$$

K_{1000} is correct to six decimal places; if one is confined to the trapezoidal rule in evaluating $\int_1^2 \exp(-y^2)\, dy$ and $\int_1^3 \exp(-x^2)\, dx$, it will require about the same number of evaluations (of a less complicated function) to achieve this accuracy.

EXAMPLE 4.2. Let $S = \{(x, y)\colon x^2 + y^2 \leqslant 1\}$. There is an extra complication in this problem in that S contains the origin, at which point v is undefined. This difficulty is circumvented by removing from S the slit $R = [-1, 10^{-6}] \times [-10^{-6}, 10^{-6}]$ and integrating over the incised region $S\backslash R$. (The error thus introduced is, by (3.7), dominated by $2 \cdot 10^{-6}$.) Parametrizing the boundary of $S\backslash R$ in a straightforward fashion, we obtain the following results from (4.1):

$$
\begin{array}{ll}
N = 8 & K_N = .172523(1) \\
N = 40 & K_N = .197425(1) \\
N = 200 & K_N = .198539(1) \\
N = 1000 & K_N = .198584(1)
\end{array}
$$

The correct value to six significant figures is 1.98586.

EXAMPLE 4.3. Let S be the region between the curves $x + y = 5.2$ and $xy = 1$. It is with such irregularly-shaped regions that the method (4.2) begins to pay off, in that all that is required is a parametrization of the boundary, and no transformation to a "standard" region is needed. (4.1) produces the following:

$N = 8$	$K_N = .399689(-2)$
$N = 40$	$K_N = .472084(-2)$
$N = 200$	$K_N = .485760(-2)$
$N = 1000$	$K_N = .488594(-2)$

In the above examples, we have been concentrating on the special case, $\phi(t) = t$, of Theorem 3.2. Other choices of ϕ may be put to good use in corroborating results already obtained. Our last example is a rerun of Example 4.1 with $\phi_i(t)$ chosen to be $t^2/2$ and $\ln t$, respectively. With the indirect aid of Lemma 3.5, we find corresponding functions w_i to be $\frac{1}{2}\exp(x^2 + y^2)a(x, y)$ and $\frac{1}{2}\exp(-x^2 - y^2)a(x, y)$. That is, let $\phi_1(t) = t^2/2$ and $w_1 = v/u$; then $[\partial(u, w_1)/\partial(x, y)]\ \phi_1'(u) = u$, and (3.4) implies that

$$\iint_S \exp(-x^2 - y^2)\, dx\, dy = \frac{1}{2}\int_T u^2\, dw_1 . \tag{4.4}$$

Likewise, for $\phi_2(t) = \ln\ t$ and $w_2 = uv$, it is found that $[\partial(u, w_2)/\partial(x, y)]\ \phi_2'(u) = u$, whence

$$\iint_S \exp(-x^2 - y^2)\, dx\, dy = \int_T \ln u\, dw_2 . \tag{4.5}$$

The right hand sides of (4.4) and (4.5) will be respectively approximated by

$$L_N = \frac{1}{4}\sum_{i=0}^{N-1}\left(\frac{v_{i+1}}{u_{i+1}} - \frac{v_i}{u_i}\right)(u_i^2 + u_{i+1}^2) \tag{4.6}$$

and

$$M_N = \frac{1}{2}\sum_{i=0}^{N-1}(u_{i+1}v_{i+1} - u_iv_i)\ln(u_iu_{i+1}). \tag{4.7}$$

EXAMPLE 4.4. Let $S = [1, 3] \times [1, 2]$, as in Example 4.1. (4.6) and (4.7) yield:

$N = 8$	$L_N = .119334$	$M_N = .139571(-1)$
$N = 40$	$L_N = .208619(-1)$	$M_N = .186313(-1)$
$N = 200$	$L_N = .189307(-1)$	$M_N = .188436(-1)$
$N = 1000$	$L_N = .188557(-1)$	$M_N = .188522(-1)$

5. Notes

The application of the classical theory of differential equations to quadrature is not a new idea. The one-dimensional analogue of the work in Section 2 can be found in Chapter 2.1 of [5]. Green's Theorem was used in [7] and [8] to develop an approximation of a double integral by means of a boundary integral. The identity (2.5) differs in that the representation is now exact, but at the substantial cost of imposing the hypothesis that the integrand satisfy a given partial differential equation. The examples of Section 2, while amusing, are of little practical use, although (2.7) is recognized to be essentially Green's second identity [4, p. 133].

The objective of creating dimension-reducing procedures for arbitrary multiple integrals, while unrealistic without some sacrifice of generality, is a noble one. (See, for example, [2, p. 313], [6], [10, Section 4], and [11] for discussions of the effect of dimension on computation.) In Sections 3 and 4, the integrand $\exp(-x^2 - y^2)$ is studied with some success; this problem had been treated in [3]. (Given the integrand $u = (2\pi\sigma_x\sigma_y)^{-1}\exp[-\frac{1}{2}(x^2/\sigma_x{}^2 + y^2/\sigma_y{}^2)]$, the modification required of our method is to replace v in (4.3) by $v = \sigma_x\sigma_y a(x/\sigma_x, y/\sigma_y)$.)

In [1a] some dimension reducing is accomplished for integrals of analytic and harmonic functions over special regions.

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